1. Overview

2. Definition

3. Orbits

Geometric Satake Theorem

$$\mathrm{Sat}_G = \mathrm{Perv}_{G_O}(\mathrm{Gr}_G, \bar{\mathbb{Q}}_l) \cong \mathrm{Rep}(\hat{G}, \bar{\mathbb{Q}}_l)$$

G defined over $k = \bar{k}$.

$F: \mathcal{A} \to \oplus H^i(\mathrm{Gr}_G, \mathcal{A})$

$\tilde{G} = \underline{\mathrm{Aut}}^{\otimes}(F)$

① semi-simple $\Rightarrow$ $\tilde{G}$ reductive

② $F: \mathrm{Sat}_G \longrightarrow \mathrm{Vect}_{\bar{\mathbb{Q}}_l}$

$\mathrm{Sat}_G \searrow$ $X_\cdot(T)$-graded v.s. $\nearrow \mathrm{Vect}_{\bar{\mathbb{Q}}_l}$

$\hat{T} = \tilde{T} \to \tilde{G}$

$G: (X^\cdot, X_\cdot, \Delta, \Delta^\vee)$

$\hat{G}: (X_\cdot, X^\cdot, \Delta^\vee, \Delta)$

eg. $\widehat{GL_n} = GL_n$

$\widehat{SL_n} = PGL_n$

$\widehat{SO_{2n}} = SO_{2n}$

$\widehat{SO_{2n+1}} = Sp_{2n}$

③ $\mathbb{Q}^\vee$, $X_\cdot(T)^+$,

$\Downarrow$ simple roots, $\Downarrow$ $W = \{\text{chamber}\}$

$W \cdot (\text{simple roots}) = \text{all roots}$

④ k not algebraic closed, $\mathrm{Gal}(\bar{k}/k) =: \Gamma$

$${}^L G^{geo} = \hat{G} \rtimes_{geo} \Gamma$$

$$\mathrm{Perv}_{G_O}(\mathrm{Gr}_G) \cong \mathrm{Rep}({}^L G^{geo})$$

$G = T = \mathbb{G}_m^n$, $\quad F = k((t))$, $\mathcal{O} = k[[t]]$.

$$G_F = (k((t))^\times)^n, \quad G_{\mathcal{O}} = (k[[t]]^\times)^n,$$

$$Gr_G = \mathbb{Z}^n = X_\cdot(T)$$

$\mathrm{Perv}(Gr_G) = X_\cdot(T)$ - graded vector space

F: forget graded structure

$$\hat{T} = \underline{\mathrm{Aut}}^{\otimes}(F) = \mathbb{G}_m^n$$

$$\alpha \in X_\cdot(T), \Rightarrow \alpha \in X^\cdot(\hat{T})$$

$$\alpha : \hat{T} \longrightarrow \mathbb{G}_m \curvearrowright V_\alpha.$$

$$Gr_G = \coprod_{\lambda \in X_\cdot(T)} S_\lambda, \qquad B,\ T = B/[B,B]$$

$$Gr_T \xleftarrow{q} Gr_B \xrightarrow{i} Gr_G$$

$$S_\lambda = i(q^{-1}(\lambda))$$

$$F_\lambda = H_c^*(S_\lambda, -)$$

$$F = \bigoplus_\lambda F_\lambda : \mathrm{Sat}_G \to X_\cdot(T)\text{ - graded v.s.}$$

Affine Grassmannian $\quad G/k, \quad F = k((t)),\ \mathcal{O} = k[[t]]$.

GL_n: lattice in F^n: $\mathcal{O}$- subspace Λ s.t. $\Lambda \otimes_{\mathcal{O}} F = F^n$.

$g \in GL_n(F) = GL(F^n)$,

$g\mathcal{O}^n$ are lattices.

$g \in GL_n(\mathcal{O})$, $g\mathcal{O}^n = \mathcal{O}^n$.

$\{\text{lattices}\} = \underline{GL_n(F)/GL_n(\mathcal{O})}$

① Gr_G sheaf over $(\mathrm{Aff}_k)_{fpqc}$ $\quad R_{\mathcal{O}} = R[[t]]$, $R_F = R((t))$

R k-algebra

$Gr_G(R) = \{$ f.g. proj. $R_{\mathcal{O}}$ submodule of R_F^n that spans R_F^n $\}$

$R_1 \to R_2$ faithfully flat.

$Gr_G(R_1) \to Gr_G(R_2) \rightrightarrows Gr_G(R_2 \otimes_{R_1} R_2)$

② $Gr_G = G_F / G_{\mathcal{O}}$

$G_F(R) = G(R_F)$, $\quad G_{\mathcal{O}}(R) = G(R_{\mathcal{O}})$

$G \hookrightarrow GL_n$, $\quad Gr_G \hookrightarrow Gr_{GL_n}$

$Gr^{(N)}(R) = \{\Lambda$ lattice. $\exists N$, s.t. $t^N R_{\mathcal{O}}^n \subset \Lambda \subset t^{-N} R_{\mathcal{O}}^n\}$

$Gr = \text{"}\varinjlim\text{"}\ Gr^{(N)}$

$\downarrow$

$R_{\mathcal{O}}$-quotient of $t^{-N}R_{\mathcal{O}}^n / t^N R_{\mathcal{O}}^n = R^{2nN}$

projective over R

$$\begin{tikzcd}
Gr^{(N)} \arrow[r, hook] \arrow[d] & Gr(2nN) \arrow[d] \\
Gr^{(N+1)} \arrow[r, hook] & Gr(2n(N+1))
\end{tikzcd}$$

$\Rightarrow Gr$ is ind-projective scheme

eg. $G = \mathbb{G}_a$, $\quad Gr_G(R) = R((t))/R[[t]]$

$Gr_{\mathbb{G}_a} = \text{"}\varinjlim\text{"}\ \mathbb{A}^N$

$\underline{G_{\mathcal{O}}(R) = G(R[[t]])} = \varprojlim G(R[t]/t^n)$

$(\mathbb{G}_a)_{\mathcal{O}} = \mathrm{Spec}\ k[x_1, x_2, \cdots]$

Prop $X(R[t]/t^n)$ is represented by a scheme

$G_F(R) = G(R((t))) = \text{"}\varinjlim\text{"}\ G(t^{-n}R[[t]])$

is represented by an ind-scheme.

Third definition

$Gr_G(R) = \{(\mathcal{E}, \beta) : \mathcal{E}$ is a G-torsor on D_R, $\beta : \mathcal{E}|_{D_R^*} \to \mathcal{E}^0|_{D_R^*}$ is an isomorphism $\}$

$D_R = \mathrm{Spec}\ R[[t]]$,

$D_R^* = \mathrm{Spec}\ R((t))$.

X a curve, $x \in X$ smooth point,

$Gr_G(R) = \{(\mathcal{E}, \beta) : \mathcal{E}$ is a G-torsor on X_R, $\beta : \mathcal{E}|_{X_R^*} \to \mathcal{E}^0|_{X_R^*}$ is an isomorphism $\}$

Beauville - Laszlo Theorem

a trivialization on D_R^* can be extended to X_R^*.

X smooth curve

$Gr_{G,X}(R) = \{(x, \mathcal{E}, \beta) : x \in X(R), \cdots \}$

Beilinson - Drinfeld Grassmannian

① $G_{\mathcal{O}} = L^+G$, $G_F = LG$ ② $L^-G(R) = G(R[t^{-1}]) \leq LG$. ③ $U_F = LU$

$U = [B,B]$,

$U^- = [B^-, B^-]$

① Cartan decomposition

$$G(F) = \coprod_{\lambda \in X_\cdot(T)^+} G(\mathcal{O})\, t^\lambda\, G(\mathcal{O})$$

$$\lambda \in X_\cdot(T), \Rightarrow \mathbb{G}_m \to T \Rightarrow \begin{array}{ccc} F^\times & \to T(F) & \hookrightarrow G(F) \\ \Psi & & \\ t & \mapsto t^\lambda. & \end{array}$$

$$\Rightarrow (\mathbb{G}_m)_F \to G_F$$

abuse of notation: $t^\lambda \in G_F$, $t^\lambda \in Gr_G$.

$$Gr_\lambda = G_{\mathcal{O}} \cdot t^\lambda \subseteq Gr_G.$$

$(\mathcal{E}, \beta) \in Gr(k)$, $inv(\beta) \in G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}) = X_\cdot(T)^+$.

$$G_{\mathcal{O}} / \underbrace{G_{\mathcal{O}} \cap t^\lambda G_{\mathcal{O}} t^{-\lambda}}_{\mathrm{Stab}_{G_{\mathcal{O}}}(t^\lambda)} \xrightarrow{\sim} Gr_\lambda$$

$$\dim Gr_\lambda = \dim T_e (G_{\mathcal{O}} / G_{\mathcal{O}} \cap t^\lambda G_{\mathcal{O}} t^{-\lambda}) = \dim \mathfrak{g}_{\mathcal{O}} / \mathfrak{g}_{\mathcal{O}} \cap Ad_{t^\lambda} \mathfrak{g}_{\mathcal{O}}$$

$$= \sum_{\langle \alpha, \lambda \rangle > 0} \langle \alpha, \lambda \rangle = \langle 2\rho, \lambda \rangle.$$

$$Ad_{t^\lambda}(\mathfrak{g}_{\alpha,\mathcal{O}}) = t^{\langle \alpha, \lambda \rangle} \mathfrak{g}_{\alpha,\mathcal{O}}$$

P_λ = Parabolic group generated by U_α, $\langle \alpha, \lambda \rangle \leq 0$.

ev: $G_{\mathcal{O}} \to G$

induce $Gr_\lambda \to G/P_\lambda$

$$\begin{array}{ccccccccc} 1 & \to & L^{>0}G & \to & G_{\mathcal{O}} & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & K & \to & Gr_G & \to & G/P_\lambda & \to & 1 \end{array}$$

$L^{>0}G$ is unipotent

Cor when Gr_λ is proper, then $Gr_\lambda = G/P_\lambda$.

Proposition $Gr_{\leq \lambda} = \coprod_{\mu \leq \lambda} Gr_\mu$ is closed $= \overline{Gr_\lambda}$.

X space, $\beta \in Gr(X)$, $X_{\leq \lambda} = \{x \in X : inv_x(\beta) \leq \lambda\}$ is closed.

$\beta = id \in G(Gr)$

$ev. \ G_{\mathcal{O}} \to G$ $\quad I \backslash G_F / I = \quad X_*(T) \rtimes W$

$ev^{-1}(B) = I \to B$ (with $B \subset G$)

Construct a curve $\overset{C_{\lambda,\alpha^\vee}}{Gr_{\lambda-\alpha^\vee}} \subset \overline{Gr_\lambda}$.

[Zhu, P25]

② L^-G $\quad Gr^\lambda = L^-G \cdot t^\lambda$

Birkhoff decomposition $\quad G(F) = \bigsqcup_{\lambda \in X_*(T)^+} G(k[t^{-1}]) t^\lambda G(\mathcal{O})$.

$\overline{Gr^\lambda} = \bigsqcup_{\mu \geq \lambda} Gr^\mu$

$\underline{Gr_\mu \cap Gr^\lambda \neq \emptyset \iff \mu \geq \lambda}$

Gr^0 is open in Gr

$L^{<0}G = \ker(L^-G \to G)$, $\quad L^{<0}G \times L^+G \to LG$ is open embedding.

$\operatorname{codim} Gr^0 = 1$, $\quad \Theta = Gr \backslash Gr^0$ is a Cartier divisor

③ U_F-orbit $\quad S_\lambda = U_F \cdot t^\lambda$. $\quad Gr_T \xleftarrow{q} Gr_B \xrightarrow{i} Gr_G$

$\lambda \in X_*(T)$ $\quad = i(q^{-1}(\lambda))$

$U \triangleleft B$, $\quad S_\lambda = t^\lambda \cdot S_0$, $\quad S_0 = U_F \cdot e$

$\overline{S_\lambda} = S_{\leq \lambda}$

Another description of S_λ

$2\rho^\vee$ = sum of positive coroots

$2\rho^\vee : \mathbb{G}_m \to T \to G$ induce an action on Gr_G.

Prop $\quad S_\lambda = \{ x \in Gr_G : \lim_{s \to 0} 2\rho^\vee(s) \cdot x = t^\lambda \}$

Pf. $\quad x \in U_F$, $\quad \lim_{s \to 0} 2\rho^\vee(s) \, x \, 2\rho^\vee(-s) \in G_{\mathcal{O}}$.

$\Rightarrow$ LHS $\subseteq$ RHS $\quad \Rightarrow \checkmark$

Cor $\quad S_\lambda \cap Gr_\mu \neq \emptyset \Rightarrow t^\lambda \in \overline{Gr_\mu} \iff S_\lambda \cap \overline{Gr_\mu} \neq \emptyset$

Next Time : Serre tree

$S_{<\lambda}$ is the intersection of $S_{\leq\lambda}$ and a hyperplane.

Note $S_0 \subseteq Gr^0$, $S_{<0} \cap Gr^0 = \phi$.

$S_{<0} = S_0 \cap \Theta$